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SOME EXAMPLES OF RADIAL FLUID MOTION

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SOME EXAMPLES OF RADIAL FLUID MOTION

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1. *Introduction*

This note is concerned with problems analogous to the collapse of a spherical bubble in water or the expansion of a spherical cavity due to the pressure of an included gas, as in a submarine explosion. In the case where the cavity or bubble is situated in liquid extending to infinity, an elegant solution in terms of surface harmonics of zero order is given by Lamb (1932). We consider here the more general case where the liquid is bounded by concentric spherical surface where also the inner surface may be collapsible like a bubble or be a cavity containing gas, and further the boundaries are subject to pressure, and the liquid is subject to the action of external forces. This general case can be treated directly by using the Eulerian equations of motion as given by Ramsey (1935). But we show that these problems can be equally well dealt with by the method given by Lamb. In fact, this note arose out of an attempt, during the course of class lectures, to show that all the exercises of this type in the text-books of Milne-Thomson (1938, 14 Examples in Ex. XV, p. 431) and Ramsey (20 examples, pp. 33–39) could be solved by the same method. We also show that it can be applied to the case of liquid bounded by cylindrical surfaces.

2. *Method of Solution*

In the case of liquid bounded by spherical surfaces, if a, b be the initial radii of the inner and outer surfaces and R, R′ their values at time t, the velocity potential ϕ is given by $\phi = \mathrm{B}(t)/r$, and the boundary conditions at the surfaces, *viz.*, $-\partial\phi/\partial r = \dot{\mathrm{R}}, \dot{\mathrm{R}}'$ for $r = \mathrm{R}, \mathrm{R}'$ respectively are satisfied by taking $\mathrm{B} = \mathrm{R}^2\dot{\mathrm{R}}$ which is equal to $\mathrm{R}'^2\dot{\mathrm{R}}'$ as follows from the equation of continuity or from

$$\mathrm{R}'^3 - \mathrm{R}^3 = \text{const.}; \quad \mathrm{R}^2\dot{\mathrm{R}} = \mathrm{R}'^2\dot{\mathrm{R}}' \tag{1}$$

Thus

$$\phi = \mathrm{R}^2\dot{\mathrm{R}}/r = \mathrm{R}'^2\dot{\mathrm{R}}'/r \tag{2}$$

To this is added the pressure equation

$$\frac{p}{\rho} = -\tfrac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{\partial\phi}{\partial t} - \Omega(r) + \mathrm{F}(t) \tag{3}$$

and (2) and (3) are sufficient for the solution of the problems concerned. The additive function F (t) in (3) becomes determinate when the value of the pressure at some point of the fluid is given for all values of t, and also the initial conditions serve to determine the constants of integration that arise from the differential equations for R or R′.

3. *Particular Cases*

We now apply the above method to several particular cases which can be found as examples in the books by Ramsey and Milne-Thomson.

(*a*) *Fluid in the form of a spherical shell* :—

Given $p = 0$ for $r = R$, $p = p_0$ for $r = R'$; also for $t = 0$, $\dot{R} = \dot{R}' = 0$, $R' = 2nc$.

The pressure equation (3) becomes

$$\frac{p}{\rho} = -\tfrac{1}{2}\frac{R'^4\dot{R}'^2}{r^4} + \frac{R'^2\ddot{R}' + 2R'\dot{R}'^2}{r} + F(t) \tag{4}$$

Since $p = p_0$ for $r = R'$, this determines F (t) and gives

$$\frac{p - p_0}{\rho} = -\tfrac{1}{2}\frac{R'^4\dot{R}'^2}{r^4} + \frac{R'^2\ddot{R}' + 2R'\dot{R}'^2}{r} - R'\ddot{R}' - \tfrac{3}{2}\dot{R}'^2 \tag{5}$$

and the condition $p = 0$ for $r = R$ yields

$$-p_0/\rho = -\tfrac{1}{2}\frac{R'^4\dot{R}'^2}{R^4} + \frac{R'^2\ddot{R}' + 2R'\dot{R}'^2}{R} - R'\ddot{R}' - \tfrac{3}{2}\dot{R}'^2 \tag{6}$$

Multiplying by $2R'^2\dot{R}'$, integrating, and using (2) and the initial data we get

$$-\frac{2p_0}{3\rho}(R'^3 - 8n^3c^3) = \frac{R'^4\dot{R}'^2}{R} - R'^3\dot{R}'^2$$

which gives the value of $\dot{R}'$ at any time t. In particular when $R' = nc$

$$\dot{R}'^2 = \frac{14p_0}{3\rho}\cdot\frac{(n^3-1)^{1/3}}{n-(n^3-1)^{1/3}} \tag{7}$$

(*b*) *Fluid bounded by concentric spherical surfaces and the inner hollow filled by gas*:—

given $p = \Pi$ for $r = R'$, $\Omega = 0$; at $t = 0$, $p = p_1$ for $R = c_1$, and the gas obeys Boyle's law.

Using (2), the pressure equation (3) can be written in the form (4) or in the form

$$\frac{p}{\rho} = -\tfrac{1}{2}\frac{R^4\dot{R}^2}{r^4} + \frac{R^2\ddot{R} + 2R\dot{R}^2}{r} + F(t). \tag{4'}$$

The pressure of the gas when the internal radius is R is given by Boyle's law as $p_R = p_1 c_1^3/R^3$, and since this holds for all t, the substitution $p = p_R$ for $r = R$ in (4′) determines $F(t)$ and we get

$$\frac{p}{\rho} = -\tfrac{1}{2}\frac{R^4\dot{R}^2}{r^4} + \frac{R^2\ddot{R} + 2R\dot{R}^2}{r} - \ddot{R}R - \tfrac{3}{2}\dot{R}^2 + \frac{p_1 c_1^3}{\rho R^3} \tag{8}$$

The condition $p = \Pi$ for $r = R'$ gives

$$\frac{\Pi}{\rho} = -\tfrac{1}{2}\frac{R^4\dot{R}^2}{R'^4} + \frac{R^2\ddot{R} + 2R\dot{R}^2}{R'} - R\ddot{R} - \tfrac{3}{2}\dot{R}^2 + \frac{p_1 c_1^3}{\rho R^3}.$$

Multiplying by $2R^2\dot{R}$ and integrating from $R = c_1$ to $R = c_2$, we get

$$\frac{2\Pi}{3\rho}(c_2^3 - c_1^3) = \left[\frac{R^4\dot{R}^2}{R'} - R^3\dot{R}^2 + \frac{2p_1 c_1^3}{\rho}\log R\right]_1^2$$

$$= \frac{c_2^4 v^2}{R_2'} - c_2^3 v^2 + \frac{2p_1 c_1^3}{\rho}\log(c_2/c_1)$$

where $v = \dot{R}$ for $R = c_2$, and $\dot{R} = 0$ for $R = c_1$, and R_2' = value of R' when $R = c_2$.

Thus we get the value of v from

$$\tfrac{1}{2}v^2 = \frac{c_1^3}{c_2^3}\left\{\frac{1}{3}\left(1 - \frac{c_2^3}{c_1^3}\right)\frac{\Pi}{\rho} - \frac{p_1}{\rho}\log\frac{c_1}{c_2}\right\}\Big/\left(1 - \frac{c_2}{R_2'}\right) \tag{9}$$

(*c*) *Gravitating fluid between* r = a *and* r = b *at* t = 0:—

Given $p = 0$ for $r = R$ and $r = R'$ for all t;

$$\text{Here, } \Omega(r) = \int_R^r \frac{4\pi\rho\, x^2\, dx}{r} + \int_r^R \frac{4\pi\rho\, x^2\, dx}{x}$$

$$= \frac{4}{3}\pi\rho\left(r^2 - \frac{R^3}{r}\right) + 2\pi\rho(R'^2 - r^2) \tag{10}$$

and the pressure equation (3) becomes

$$\frac{p}{\rho} = -\tfrac{1}{2}\frac{R^4\dot{R}^2}{r^4} + \frac{R^2\ddot{R}}{r} - \tfrac{4}{3}\pi\rho\left(r^2 - \frac{R^3}{r}\right) - 2\pi\rho(R'^2 - r^2) + F(t) \tag{11}$$

Since $p = 0$ for $r = R$ this gives

$$F(t) = -(R\ddot{R} + \tfrac{3}{2}\dot{R}^2) + 2\pi\rho(R'^2 - R^2)$$

whence

$$\frac{p}{\rho} = -\tfrac{1}{2}\frac{R^4\dot{R}^2}{r^4} + \frac{R^2\ddot{R} + 2R\dot{R}^2}{r} - (R\ddot{R} + \tfrac{3}{2}\dot{R}^2) + 2\pi\rho(R'^2 - R^2)$$

$$- \frac{4}{3}\pi\rho\left(r^2 - \frac{R^3}{r}\right) - 2\pi\rho(R'^2 - r^2) \tag{12}$$

The condition $p = 0$ for $r = R'$ now gives

$$0 = -\tfrac{1}{2}\frac{R^4\dot{R}^2}{R'^4} + \frac{R^2\ddot{R} + 2R\dot{R}^2}{R'} - (R\ddot{R} + \tfrac{3}{2}\dot{R}^2) + 2\pi\rho(R'^2 - R^2) - \tfrac{4}{3}\pi\rho\left(R'^2 - \frac{R^3}{R'}\right) \quad (13)$$

At $t = 0$, $R = a$, $R' = b$ and $\dot{R} = 0$, so that

$$0 = \frac{a^2(\ddot{R})_0}{b} - a(\ddot{R})_0 + 2\pi\rho(b^2 - a^2) - \tfrac{4}{3}\pi\rho\left(b^2 - \frac{a^3}{b}\right)$$

which gives the value of $(\ddot{R})_0$ as

$$a(\ddot{R})_0 = -\tfrac{2}{3}\pi\rho(2a^2 - ab - b^2) \quad (14)$$

From (12) the initial pressure at any point of the fluid is given by

$$\frac{p_0}{\rho} = \frac{a^2(\ddot{R})_0}{r} - a(\ddot{R})_0 + 2\pi\rho(r^2 - a^2) - \tfrac{4}{3}\pi\rho\left((r^2 - \frac{a^3}{r}\right)$$

and using (14) this gives after simplification

$$p_0 = \tfrac{2}{3}\pi\rho^2(r - a)(b - r)\left(\frac{a+b}{r} + 1\right) \quad (15)$$

If we multiply (13) by $2R^2\dot{R}$, integrate, and use (1) and (2), we get

$$\frac{R^4\dot{R}^2}{R'} - R^3\dot{R}^2 + \frac{4\pi\rho}{5}(R'^5 - b^5) - \frac{4\pi\rho}{5}(R^5 - a^5) - \frac{4\pi\rho}{3}(R'^5 - b^5) + \frac{4\pi\rho}{3}(R^3R'^2 - a^3b^2) = 0$$

i.e.,
$$\dot{R}^2 = \frac{4\pi\rho}{15}\cdot\frac{R'\{2(R^5 - b^5) - 5(R^3R'^3 - a^3b^2) + 3(R^5 - a^5)\}}{R^3(R' - R)} \quad (15')$$

(*d*) *Pulsating sphere bounded by infinite liquid*:—

Given $R = a + b\cos nt$, $p = 0$ for $r = \infty$.

The pressure equation is given by

$$\frac{p - p_0}{\rho} = -\tfrac{1}{2}\frac{R^4\dot{R}^2}{r^4} + \frac{R^2\ddot{R} + 2R\dot{R}^2}{r}$$

and with $R = a + b\cos nt$, this gives

$$p_0 - p = \frac{\rho bn^2}{r}(a + b\cos nt)\left\{b(1 - 3\sin^2 nt) + a\cos nt + \frac{b\sin^2 nt}{2r^3}(a + b\cos nt)^3\right\}$$

At the surface of the sphere $r = R$, and the pressure there is

$$p = p_0 - \rho bn^2\{b(1 - 3\sin^2 nt) + a\cos nt + \tfrac{1}{2}b\sin^2 nt\}$$

and the least value of this is

$$p_{\min} = p_0 - \rho bn^2 (a+b). \tag{16}$$

The above are typical examples, and others can be solved by the same method.

4. *Fluid Bounded by Cylindrical Surfaces*

Let R, R′ be the inner and outer radii of the cylinders. If the velocity potential be taken as

$$\phi = \mathrm{B} \log r$$

where r is the distance from the axis, and B a function of the time, then for the boundary condition at $r = \mathrm{R}'$, $-\frac{\partial \phi}{\partial r} = \dot{\mathrm{R}}'$ for all t. Thus we choose $\mathrm{B} = -\mathrm{R}'\dot{\mathrm{R}}'$, *i.e.*,

$$\phi = -\mathrm{R}'\dot{\mathrm{R}}' \log r. \tag{17}$$

In place of (1) we have

$$\mathrm{R}'\dot{\mathrm{R}}' = \mathrm{R}\dot{\mathrm{R}}; \quad \mathrm{R}'^2 - \mathrm{R}^2 = a^2 - b^2 \tag{18}$$

The pressure equation now becomes

$$\frac{p}{\rho} = -\tfrac{1}{2}\frac{\mathrm{R}'^2\dot{\mathrm{R}}'^2}{r^2} - (\mathrm{R}'\ddot{\mathrm{R}}' + \dot{\mathrm{R}}'^2) \log r + \mathrm{F}(t) \tag{19}$$

if there be no external forces. F (t) is determined by the condition $p = \Pi$ for $r = \mathrm{R}'$ for all t, and

$$\frac{\Pi - p}{\rho} = \tfrac{1}{2}\left(\frac{\mathrm{R}'^2}{r^2} - 1\right)\dot{\mathrm{R}}'^2 - (\mathrm{R}^2\ddot{\mathrm{R}}' + \dot{\mathrm{R}}'^2) \log \frac{\mathrm{R}'}{r}. \tag{20}$$

If the internal cylinder suddenly collapses we can now find its velocity as also the initial pressure at any point of the fluid.

The condition $p = 0$ for $r = \mathrm{R}$ yields

$$\frac{\Pi}{\rho} = \tfrac{1}{2}\left(\frac{\mathrm{R}'^2}{\mathrm{R}^2} - 1\right)\dot{\mathrm{R}}'^2 - (\mathrm{R}'\ddot{\mathrm{R}}' + \dot{\mathrm{R}}'^2) \log \frac{\mathrm{R}'}{\mathrm{R}}. \tag{21}$$

Multiplying by $2\mathrm{R}\dot{\mathrm{R}}$ and using (18), we have after some simplification

$$\frac{2\Pi}{\rho}\mathrm{R}\dot{\mathrm{R}} = -\frac{d}{dt}\left(\mathrm{R}^2\dot{\mathrm{R}}^2 \log \frac{\mathrm{R}'}{\mathrm{R}}\right)$$

and integrating

$$\frac{\Pi}{\rho}\mathrm{R}^2 = -\mathrm{R}^2\dot{\mathrm{R}}^2 \log \frac{\mathrm{R}'}{\mathrm{R}} + \text{const.}$$

If for $t=0$, $R=b$, $R'=a$ the constant is determined, and

$$R^2\dot{R}^2 \log\left(\frac{R'}{R}\right)=\frac{\Pi}{\rho}(b^2-R^2) \tag{22}$$

which determines the velocity at the internal surface.

Equation (20) gives for $t=0$, if $p_0=$ initial pressure at r

$$(\Pi-p_0)/\rho=-a(\ddot{R}')_0 \log\frac{a}{r}$$

and from (21) for $t=0$,

$$\frac{\Pi}{\rho}=-a(\ddot{R}')_0 \log\frac{a}{b}.$$

Hence

$$p_0=\Pi\left\{1-\frac{\log a-\log r}{\log a-\log b}\right\}$$

$$=\Pi\cdot\frac{\log r-\log b}{\log a-\log b}. \tag{23}$$

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$4T+2V$

$4(h-V)+2V$

$-2V+\text{const.}$